Torsional oscillations in structures subject to ground motion

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A multiple-scale technique has been employed to study nonlinear torsional oscillations in single-storey structures with cubic softening stiffness members subject to a single frequency ground excitation. The structures are studied under free oscillation conditions, primary resonance, and combination resonance. Time history analyses are employed to quantify the dynamic behaviour of a single-storey structure. The instabilities are related to the jump in response which exists in single degree-of-freedom softening oscillators subjected to harmonic excitation. It is shown that this jump can be quantified analytically, and appears as a cusp catastrophe in the bending mode. Time history analyses show that the bending mode can exhibit a limit cycle behaviour of constant amplitude.

Key words: oscillations, earthquakes, perturbation techniques, catastrophe

It is well known that damaging torsional motion of structures is not uncommon during earthquakes. It has been suggested[1] that the rotational component of ground motion induces torsion in symmetric structures and it has been demonstrated[2,3] that linear elastic models can undergo torsional oscillation provided that the ratio of natural frequencies approaches unity. Kan and Chopra[4–6] examined elastic buildings with eccentricities to draw comparisons between the torsionally coupled and uncoupled system. In addition, they showed that torsion in multi-storey buildings can be modelled by analogy to single-storey structures.

Nonlinear coupling has been employed to identify regions of instability. Torsional instabilities in a single-storey symmetrical structure having a cubic softening stiffness relationship have been identified.[7] Antonelli *et al.*[8] reported a more general analysis for multi-storey unsymmetrical structures, of which the model of reference 7 is a special case. Syamal and Pekau[9] studied instabilities in single-storey unsymmetrical structures. The above three papers all treated the case of cubic softening using different mathematical approaches and produced findings consistent with one another. Basically, they identified instability regions, as plotted on a space of forcing frequency *versus* bending frequency *versus* torsional frequency. Tso[7] directly obtained a Mathieu equation and wrote a (first term) series approximation to the solution by inspection. In reference 8 a variational method was used which produced a Mathieu equation for the single-storey case or a higher order determinantal equation for multi-storey structures. Syamal and Pekau[9] employed the Krylov–Bogoliubov[10] method with comparable results.

All previous work carried the analytical solution only as far as identifying the instability regions, as typically defined when linear variations about the equilibrium state were indeterminate. Antonelli *et al.*[8] presented some time histories and concluded that instability was manifest as a breakdown in the conventional steady state response. Syamal and Pekau[9] experienced 'gaps' in their solution, as did Evenson[11] in his study of ring dynamics. None of the previous efforts contained an analysis of the response form within the instability regions.

In a recent paper, Ramos[12] employed a two-time perturbation technique to study the lateral and torsional motions of a nonlinear symmetrical structure subject to a lateral sinusoidal motion, and showed that the structure is particularly susceptible to torsional oscillations when the ground acceleration frequency is about one-third of or three times larger than the natural frequency of the lateral motion. In this paper, the same symmetrical single-storey structure with cubic softening members and subject to the same ground excitation is analysed by means of a two-time perturbation technique to assess the effects of weak non-

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linearities on the bending and torsional responses. Analytic solutions are obtained for the free oscillations, and primary and combination resonances of the structure. It is shown that the amplitude of the bending mode presents a cusp catastrophe when plotted *versus* the excitation frequency and damping coefficients. It is also shown that instabilities occur as jump phenomena in the bending mode and that the torsional mode amplitude is a monotonically decreasing function of time.

Problem formulation

Figure 1 reproduces the structure model.[7,8,12] A ground acceleration in the u-direction of $-A\cos\omega t$ is applied, and all columns have a force-displacement relationship of the form $F = k[1-\lambda(d/d_0)^2]d$ where d is the column displacement, d_0 is a reference displacement, k is the initial tangent stiffness, and λ is the softening coefficient. The fundamental frequencies for motion in the x, y, and θ-directions, under linear conditions, are denoted as ω_x, ω_y, and ω_θ, respectively. The coupled equations of motion in dimensional form are:[7,8,12]

$$u_1'' + 2\zeta_x\omega_x u_1' + \omega_x^2 u_1 = -A^*\cos\omega^* t^* + \epsilon^*\omega_x^2 u_1 u_1^2 + 3b^2\theta_1^2)$$

$$\theta_1'' + 2\zeta_\theta\omega_\theta\theta_1' + \omega_\theta^2\theta_1 = \epsilon^*\theta_1(\alpha_1 u_1^2 + \alpha_2\theta_1^2)$$

where the primes denote differentiation with respect to dimensional time t^* and where:

$$\epsilon^* = \lambda/d_0^2 \quad r = a/b \quad \alpha_1 = 9\omega_x^2/(1+r^2)$$

$$\alpha_2 = 3(\omega_x^2 b^4 + \omega_y^2 a^4)/(a^2+b^2)$$

Upon making the following substitution:

$$\omega_1^* = \omega_x \quad \omega_2^* = \omega_\theta \quad a_1^{**} = 2\zeta_x\omega_x \quad a_2^* = \epsilon^*\omega_x^2$$

$$a_3^* = 3\epsilon^*\omega_x^2 b^2 \quad a_4^{**} = 2\zeta_\theta\omega_\theta$$

$$a_5^* = \epsilon^*\alpha_1 \quad a_6^* = \epsilon^*\alpha_2$$

the governing equations can be written as:

$$u_1'' + a_1^{**}u_1' + \omega_1^{*2}u_1 = a_2^* u_1^3 + a_3^* u_1\theta_1^2 - A^*\cos\omega^* t^*$$

$$\theta_1'' + a_4^{**}\theta_1' + \omega_2^{*2}\theta_1 = a_5^* u_1^2\theta_1 + a_6^*\theta_1^3$$

These equations can be non-dimensionalized by using appropriate values for u, θ and t^*. In the present study, t^* has been made dimensionless with respect to the natural bending frequency ω_1 by the transformation $t = \omega_1 t^*$, where t is the dimensionless time variable, so that the dimensionless equations can be written as:

$$u'' + a_1^* u' + u = a_2 u^3 + a_3 u\theta^2 - A\cos\omega t \qquad (1)$$

$$\theta'' + a_4^*\theta' + \omega_2^2\theta = a_5 u^2\theta + a_6\theta^3 \qquad (2)$$

where, for example, $u = u_1/u_0$, $\omega = \omega^*/\omega_1$, $\theta = \theta_1/\theta_0$ and $a_2 = a_2^* u_0^2/\omega_1^2$, and u_0 and θ_0 are the dimensional bending and torsional displacements used in the non-dimensionalization. In the calculation reported here the following values have been employed:

$$a_1 = 0.5 \quad a_2 = 1 \quad a_3 = 1 \quad a_4 = 0.01$$

$$a_5 = 1 \quad a_6 = 1 \quad F = 100 \quad \epsilon = 0.1$$

The values of ω_2 and ω are varied in this study according to the phenomenon under consideration.

One of the main interests in this paper is to analyse the response in θ that can be generated as a result of the nonlinearities. The main interest is in weak nonlinearities, particularly when the damping, nonlinear terms and excitation are of the same order of magnitude. For each nonlinearities a small parameter, ϵ, is introduced, such that the values of a_1^*, a_4^* and A become:

$$a_1^* = \epsilon^2 a_1 \quad a_4^* = \epsilon^2 a_4 \quad A = \epsilon^3 F$$

Then equations (1) and (2) can be written as:

$$u'' + u = a_2 u^3 + a_3 u\theta^2 - \epsilon^2 a_1 u' - \epsilon^3 F\cos\omega t \qquad (3)$$

$$\theta'' + \omega_2^2\theta = a_5 u^2\theta + a_6\theta^3 - \epsilon^2 a_4\theta' \qquad (4)$$

where primes denote differentiation with respect to t.

Analytical methodology

For weak nonlinearities one looks for a solution of the form:

$$u = \epsilon u_1 + \epsilon^3 u_2 + O(\epsilon^5) \qquad (5)$$

$$\theta = \epsilon\theta_1 + \epsilon^3\theta_2 + O(\epsilon^5) \qquad (6)$$

Moreover, introducing two time scales: $T_0 = t$ and $T_2 = \epsilon^2 t$, and substituting equations (5) and (6) into (3) and (4):

$$\epsilon(D_0^2 u_1 + u_1) + \epsilon^3(2D_0D_2u_1 + D_0^2u_2 + u_2 - a_2u_1^3 + F\cos\omega t - a_3u_1\theta_1^2 + a_1D_0u_1) = 0 \qquad (7)$$

and

$$\epsilon(D_0^2\theta_1 + \omega_2^2\theta_1) + \epsilon^3(2D_0D_2\theta_1 + D_0^2\theta_2 + \omega_2^2\theta_2 - a_5u_1^2\theta_1 - a_6\theta_1^3 + a_4D_0\theta_1) = 0 \qquad (8)$$

where $D_0 = \partial/\partial T_0$ and $D_2 = \partial/\partial T_2$, and where terms of $O(\epsilon^5)$ have been neglected.

Collection of terms in ϵ yields:

$$D_0^2u_1 + u_1 = 0 \qquad (9)$$

and

$$D_0^2\theta_1 + \omega_2^2\theta_1 = 0 \qquad (10)$$

Collection of terms in ϵ^3 yields:

$$D_0^2u_2 + u_2 = a_2u_1^3 - F\cos\omega t + a_3u_1\theta_1^2 - a_1D_0u_1 - 2D_0D_2u_1 \qquad (11)$$

$$D_0^2\theta_2 + \omega_2^2\theta_2 = a_5u_1^2\theta_1 + a_6\theta_1^3 - a_4D_0\theta_1 - 2D_0D_2\theta_1 \qquad (12)$$

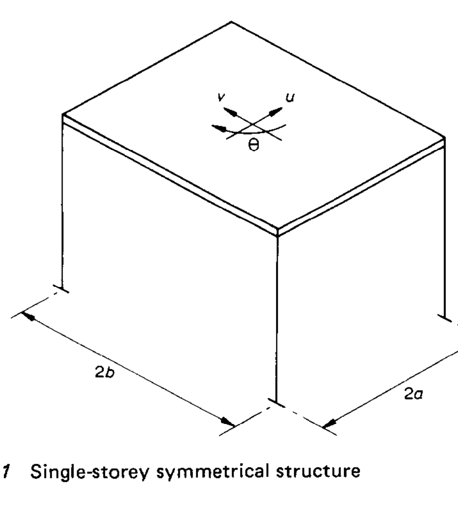

Figure 1 Single-storey symmetrical structure

Equations (9)–(12) form a system of differential equations for the displacements u_1, u_2, θ_1 and θ_2. Specifically, equations (9) and (10) can be solved directly to yield:

$$u_1 = A_1 \exp[iT_0] + \mathrm{cc} \tag{13}$$

$$\theta_1 = A_2 \exp[i\omega_2 T_0] + \mathrm{cc} \tag{14}$$

where A_1 and A_2 are functions of T_2, and where cc stands for the complex conjugate of the previous terms.

Noting that $F \cos \omega t$ can be written as:

$$(F/2)(\exp[i\omega T_0] + \exp[-i\omega T_0])$$

equations (13) and (14) may be substituted into (11) and (12) to yield lengthy equations which reduce to equations for u_2 and for θ_2 as follows:

$$\begin{aligned} D_0^2 u_2 + u_2 &= -(F/2)\exp[i\omega T_0] + a_2 A_1^3 \exp[3iT_0] \\ &\quad + 3a_2 A_1^2 \bar{A}_1 \exp[iT_0] - ia_1 A_1 \exp[iT_0] \\ &\quad + a_3 A_1 A_2^2 \exp[i(1+2\omega_2)T_0] \\ &\quad + a_3 \bar{A}_1 A_2^2 \exp[i(-1+2\omega_2)T_0] - 2i\frac{dA_1}{dT_2}\exp[iT_0] \\ &\quad + 2a_3 A_1 A_2 \bar{A}_2 \exp[iT_0] + \mathrm{cc} \end{aligned} \tag{15}$$

$$\begin{aligned} D_0^2 \theta_2 + \omega_2^2 \theta_2 &= -2i\omega_2 \frac{dA_2}{dT_2}\exp[i\omega_2 T_0] - a_4 i\omega_2 A_2 \exp[i\omega_2 T_0] \\ &\quad + a_5 A_1^2 A_2 \exp[i(2+\omega_2)T_0] \\ &\quad + a_5 A_1^2 \bar{A}_2 \exp[i(2-\omega_2)T_0] \\ &\quad + 2a_5 A_1 \bar{A}_1 A_2 \exp[i\omega_2 T_0] + a_6 A_2^3 \exp[3i\omega_2 T_0] \\ &\quad + 3a_6 A_2^2 \bar{A}_2 \exp[i\omega_2 T_0] + \mathrm{cc} \end{aligned} \tag{16}$$

where an overbar indicates complex conjugate.

Equations (15) and (16) have homogeneous solutions proportional to $\exp[iT_0]$ and $\exp[-iT_0]$. The right-hand side of these equations contains terms proportional to the homogeneous solutions; hence, they exhibit a secular character. In addition, if ω approaches unity there is resonance as the excitation frequency approaches the natural frequency of the bending mode. Equation (15) also exhibits a secular character whenever ω_2 approaches unity. The behaviour exhibited by equation (15) can be classified as follows:

(1) primary resonance as $\omega \sim 1$
(2) internal resonance as $\omega_2 \sim 1$
(3) combination resonance as $\omega \sim 1$ and $\omega_2 \sim 1$

Equation (16) also contains secular terms proportional to $\exp[i\omega_2 T_0]$. It also exhibits internal resonance whenever ω_2 approaches unity. The free response as well as the responses under primary and combination resonances are subsequently studied.

Free response

For the free response case F is zero, and ω_2 is removed from unity. Equations (15) and (16) exhibit a secular character unless the terms proportional to $\exp[iT_0]$ and $\exp[i\omega_2 T_0]$ are set to zero. Eliminating secular terms, one obtains:

$$3a_2 A_1^2 \bar{A}_1 - ia_1 A_1 - 2i\frac{dA_1}{dT_2} + 2a_3 A_1 A_2 \bar{A}_2 = 0 \tag{17}$$

and

$$-2i\omega_2 \frac{dA_2}{dT_2} - i\omega_2 a_4 A_2 + 2a_5 A_1 \bar{A}_1 A_2 + 3a_6 A_2^2 \bar{A}_2 = 0 \tag{18}$$

A solution of the form:

$$\begin{aligned} A_1 &= a \exp[i\beta] \\ A_2 &= \gamma \exp[i\delta] \end{aligned} \tag{19}$$

can be assumed where α, β, γ, and δ are time-dependent variables. Substituting equations (18) and (19) into (16) and (17) one obtains:

$$\begin{aligned} &3a_2\alpha^3 + 2\alpha\beta' + 2a_3\alpha\gamma^2 = 0 \\ &a_1\alpha + 2\alpha' = 0 \\ &2\omega_2\gamma\delta' + 2a_5\alpha^2\gamma + 3a_6\gamma^3 = 0 \\ &2\omega_2\gamma' + a_4\omega_2\gamma = 0 \end{aligned} \tag{20}$$

where a prime now denotes differentiation with respect to T_2.

The solution of equations (20) is:

$$\begin{aligned} \alpha(T_2) &= P\exp[-a_1 T_2/2] \\ \gamma(T_2) &= Q\exp[-a_4 T_2/2] \\ \beta(T_2) &= M + \frac{1}{2}\left[2\frac{a_3}{a_4}\gamma^2 + 3\frac{a_2}{a_1}\alpha^2\right] \\ \delta(T_2) &= N + \frac{1}{2\omega_2}\left[2\frac{a_5}{a_1}\alpha^2 + 3\frac{a_6}{a_4}\gamma^2\right] \end{aligned} \tag{21}$$

where P, Q, M and N are constants determined from the initial conditions.

Finally, combining equations (21), (19), (13), (14), (6) and (5) the structural response can be written as:

$$\begin{aligned} u &= 2\epsilon\alpha\cos(T_0 + \beta) + O(\epsilon^3) \\ \theta &= 2\epsilon\gamma\cos(\omega_2 T_0 + \delta) + O(\epsilon^3) \end{aligned} \tag{22}$$

Equations (22) then generate the time history response for u and θ. Note that α and γ are functions of T_2 and that they represent in this case the damping of the free vibration response. Furthermore, β and δ remain functions of T_2, and it is obvious that the responses in u and θ are of order ϵ.

Equations (20) were solved by means of a fourth-order Runge-Kutta method to yield the values of α, β, γ, and δ as a function of time. These values were then substituted into equations (22) to yield u and θ. Some sample results are shown in *Figures 2* and *3* for $\omega_2 = 4$. *Figures 1* and *2* present the time histories of u and θ. It is clear from these figures that the u-response is damped in about 25 dimensionless time units, while the θ-response is damped more slowly. All the calculations presented in this paper correspond to $u(0) = \theta(0) = 1$ and $u'(0) = \theta'(0) = 0$.

Free response with frequency incidence

For free response with frequency incidence F is still zero but ω_2 is close to 1, i.e.:

$$\omega_2 = 1 + \epsilon^2 \rho$$

where ρ is a 'detuning parameter'. Also introducing the varible Π:

$$\Pi = -\beta + \delta + \epsilon^2 \rho t$$

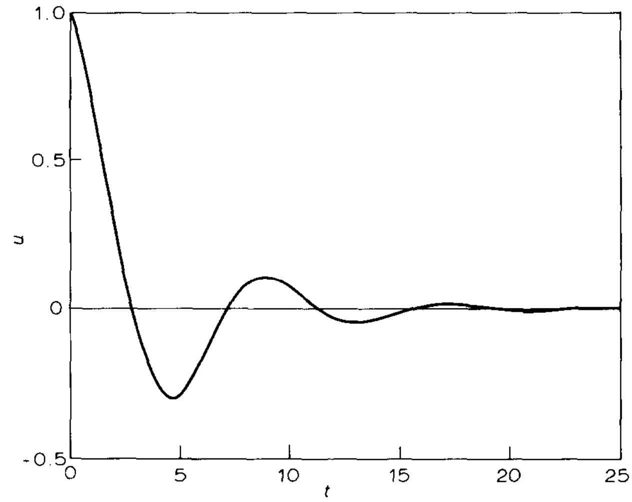

Figure 2 Bending or lateral response for zero ground excitation when bending frequency is different from torsion frequency

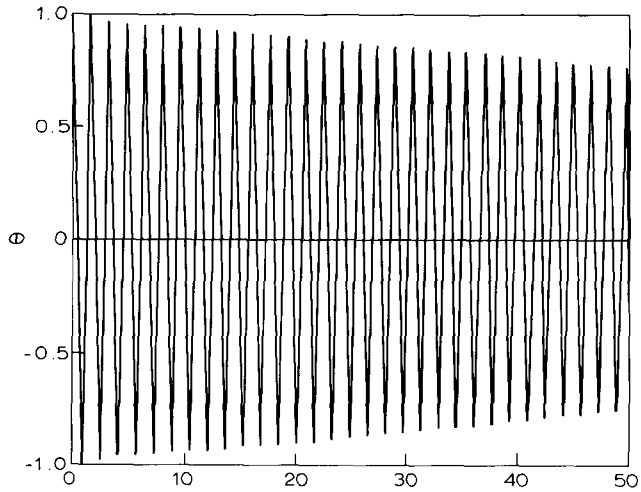

Figure 3 Torsional response for zero ground excitation when bending frequency is different from torsion frequency

Substituting equations (19) and the values of ω_2 and Π into (15) and (16), and eliminating secular terms, the governing equations for the solution variables become:

$$
\begin{aligned}
\alpha' &= -\frac{1}{2}\left[-a_3\alpha\gamma^2 \sin 2\Pi + a_1\alpha\right] \\
\gamma' &= -\frac{1}{2\omega_2}\left[a_4\omega_2\gamma + a_5\alpha^2\gamma \sin 2\Pi\right] \\
\beta' &= -\frac{1}{2\alpha}\left[3a_2\alpha^3 + 2a_3\alpha\gamma^2 + a_3\alpha\gamma^2 \cos 2\Pi\right] \qquad (23) \\
\Pi' &= -\frac{1}{2\omega_2\gamma}\left[a_5\alpha^2\gamma \cos 2\Pi + 2a_5\alpha^2\gamma + 3a_6\gamma^3\right] + \rho \\
&\quad + \frac{1}{2\alpha}\left[3a_2\alpha^3 + 2a_3\alpha\gamma^2 + a_3\alpha\gamma^2 \cos 2\Pi\right]
\end{aligned}
$$

In steady state conditions $\alpha' = \beta' = \gamma' = \Pi' = 0$ and the steady state equations admit the trivial solution $\alpha = \gamma = 0$. However, it can be shown that for non-trivial steady state response the following condition would have to apply:

$$\left(\frac{\alpha}{\gamma}\right)^2 = -\frac{a_3 a_4 \omega_2}{a_1 a_5}$$

This condition cannot be satisfied with real values of (α/γ) for positive values of a_1, a_3, a_4, and a_5, which means that a steady state response is impossible. This has been verified numerically by integrating equations (23) using a Runge-Kutta method to yield α, γ, β and δ, and then substituting these into equations (19), (13), (14), (5) and (6) to yield the response time history of u and θ. Sample histories are shown in *Figures 4* and *5*, using the same initial conditions and parameters as those in *Figures 2* and *3*, and $\rho = 10$. *Figures 4* and *5* indicate that the u-response oscillates rapidly and is higher than that observed in *Figures 2* and *3*. The θ-response is quite different from that shown in *Figures 2* and *3*. Both responses, however, eventually die out since the system is not excited by a ground acceleration.

A comparison between *Figures 2* and *4* shows that, in the case of frequency incidence (*Figure 3*), the amplitude

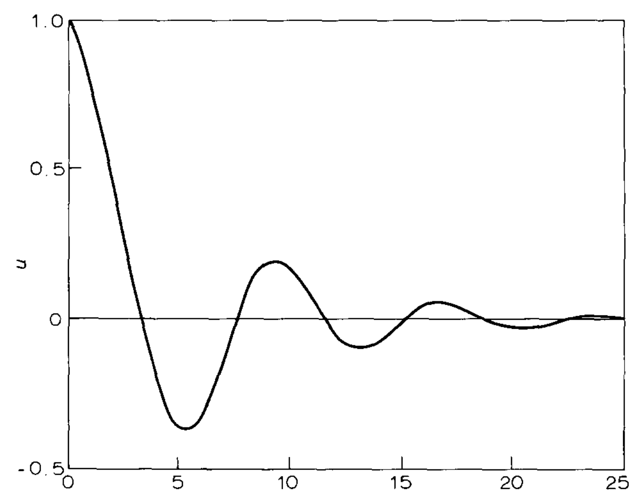

Figure 4 Bending or lateral response for zero ground excitation when bending frequency is approximately equal to torsion frequency

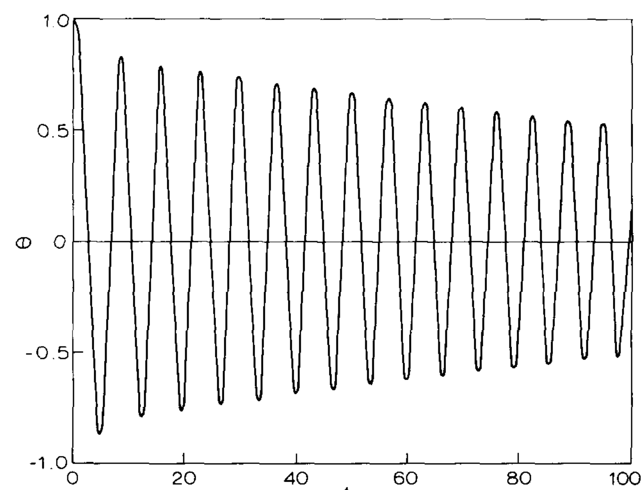

Figure 5 Torsional response for zero ground excitation when bending frequency is approximately equal to torsion frequency

of the bending mode decreases much more slowly than that of the free response in the absence of frequency incidence. *Figure 4* also indicates that the bending mode frequency is higher in the case of frequency incidence because of the transfer of energy due to nonlinearities between the bending and torsion modes.

Forced response: resonance in the absence of frequency incidence

In the case of forced response the forcing frequency approaches unity. As the excitation frequency approaches the natural undamped frequency of the linear system, primary resonance occurs. To study this phenomenon a detuning parameter, σ, is introduced, such that:

$$\omega = 1 + \epsilon^2\sigma \tag{24}$$

Introducing equations (24) and (19) into (15) and (16), and eliminating secular terms, one obtains:

$$\begin{aligned} &-\frac{F}{2}\cos\Pi + 3a_2\alpha^3 + 2\alpha\sigma - 2\alpha\Pi' + 2a_3\alpha\gamma^2 = 0 \\ &\frac{F}{2}\sin\Pi + a_1\alpha + 2\alpha' = 0 \\ &2\omega_2\gamma' + a_4\omega_2\gamma = 0 \\ &2\omega_2\gamma\delta' + 2a_5\alpha^2\delta + 3a_6\gamma^3 = 0 \end{aligned} \tag{25}$$

where $\Pi = \sigma\epsilon^2 t - \beta$.

In the steady state, $\alpha' = \Pi' = \delta' = \gamma' = 0$. Hence, in steady state $\gamma = \delta = 0$, and α and Π are given by the following equations:

$$\sigma = -\frac{3a_2}{2}\alpha^2 \pm \frac{1}{4\alpha}(F^2 - 4a_1^2\alpha^2)^{1/2} \tag{26}$$

$$\sin\Pi = -\frac{2}{F}a_1\alpha$$

Equation (26) shows that the maximum amplitude is:

$$\alpha_{\max} = \frac{F}{2a_1}$$

This maximum amplitude is associated with the following values for the detuning parameter σ and excitation frequency (*cf.* equation (24)):

$$\sigma = -\frac{3a_2F^2}{8a_1^2} \qquad \omega = 1 - \frac{3a_2F^2}{8a_1^2}\epsilon^2$$

Equation (26) shows that for a given value of σ there can be three possible values of α, where α represents the amplitude of the displacement u. A representative result is shown in *Figure 6*. If it were possible to perform a steady state experiment, a decrease in ω (from some position greater than unity) would correspond to an increase in α, until a value of ω is reached at which the tangent to the curve for α is vertical. At that point the steady state solution would jump to the lower branch. If the experiment were performed by increasing the value of ω (from some position less than unity) the amplitude would increase along the lower branch but would then jump when the vertical tangent is reached. This jump is characteristic of nonlinear equations such as Duffing's equation[12] and corresponds to a cusp catastrophe.[13] Some sample results of this cusp catastrophe are shown in *Figure 6* for several excitation forces F and damping coefficients. Curve 2 corresponds to a structure subject to a higher ground acceleration than curve 1. This is indicated by the larger bending mode amplitude. Curve 3 corresponds to a structure with larger damping, i.e. larger a_1, than curve 1. This is indicated by the lower bending mode amplitude. Thus, larger excitation forces and smaller damping coefficients yield higher peak amplitudes. Of course, different damping coefficients correspond to different structures.

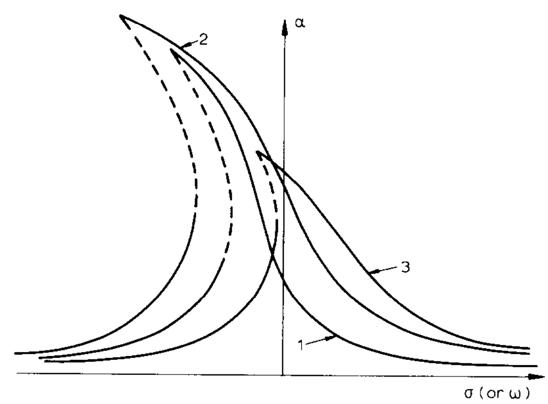

Figure 6 Steady state amplitude of bending response for non-zero ground excitation when bending frequency is different from torsion frequency

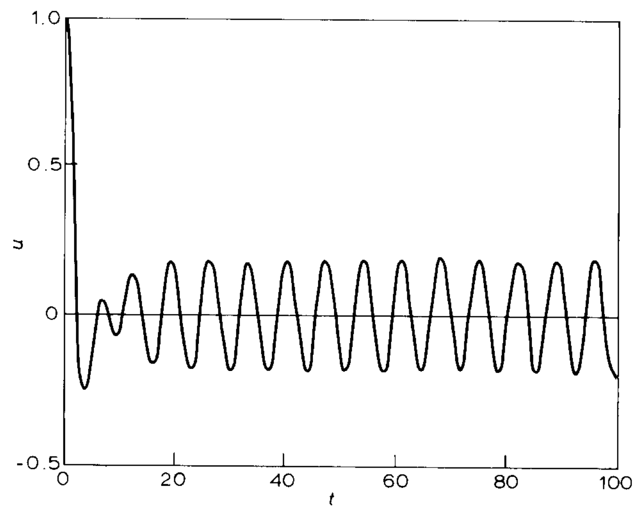

Figure 7 Bending or lateral response for non-zero ground excitation when bending frequency is approximately equal to excitation frequency but different from torsion frequency

Equations (25) were also solved numerically by means of a fourth-order Runge-Kutta method. These equations yielded the values of α, β, γ, and δ, which were then substituted into equations (19), (13), (14), (6) and (5) to obtain u and θ. Some sample results for the case $\sigma = -10$ and $\omega_2 = 4$ are presented in *Figures 7* and *8*. *Figure 7* shows the u-response and indicates that initially the u-mode is damped until approximately $t = 15$, but then starts growing and eventually reaches a steady state solution, with constant amplitude and constant frequency. In contrast, the θ-response is slowly damped (since ω_2 is removed from unity) as seen in *Figure 8*.

Forced response: resonance with frequency incidence

Next consider the case in which both ω and ω_2 approach unity. Two detuning parameters, ρ and σ, are introduced

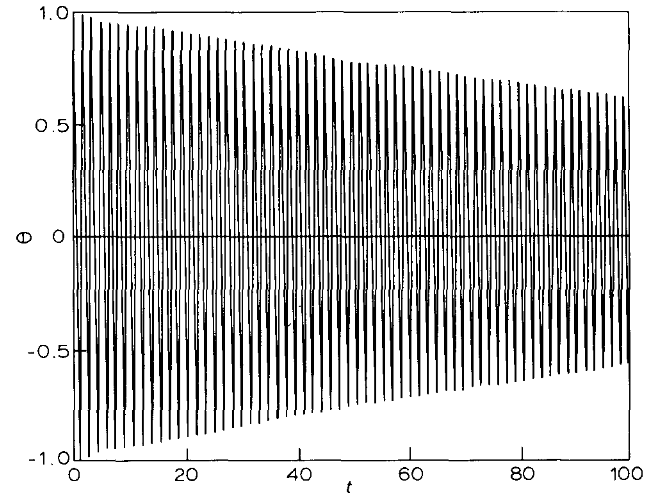

Figure 8 Torsional response for non-zero ground excitation when bending frequency is approximately equal to excitation frequency but different from torsion frequency

as follows:

$$\omega_2 = 1 - \epsilon^2\rho \qquad \omega = 1 + \epsilon^2\sigma$$

along with the variables μ and Π defined by:

$$\beta = \epsilon^2\sigma t - \mu \qquad \delta = \epsilon^2 t(\sigma + \rho) - (\Pi + \mu)$$

Substituting the variables β and δ and equations (19) into (15) and (160 and eliminating secular terms, it can be shown that:

$$\mu' = \frac{1}{2\alpha}\left[2a_3\alpha\gamma^2 + 2\alpha\sigma + a_3\alpha\gamma^2\cos 2\Pi + 3a_2\alpha^3 - \frac{F}{2}\cos\mu\right] \tag{27a}$$

$$\alpha' = -\frac{1}{2}\left[a_1\alpha + \frac{F}{2}\sin\mu + a_3\alpha\gamma^2\sin 2\Pi\right] \tag{27b}$$

$$\Pi' = -\frac{1}{2\alpha}\left[2a_3\alpha\gamma^2 + 2\alpha\sigma + a_3\alpha\gamma^2\cos 2\Pi + 3a_2\alpha^3 - \frac{F}{2}\cos\mu\right] + \frac{1}{2\omega_2\gamma}[2\omega_2\gamma(\sigma+\rho) + a_5\alpha^2\gamma\cos 2\Pi + 2a_5\alpha^2\gamma + 3a_6\gamma^3] \tag{27c}$$

$$\gamma' = -\frac{1}{2\omega_2}[a_4\omega_2\gamma - a_5\alpha^2\gamma\sin 2\Pi] \tag{27d}$$

In steady state $\alpha' = \Pi' = \mu' = \delta' = 0$ and equations (27) can be combined and written as:

$$\frac{F^2}{4\alpha^2} = \left[3a_2\alpha^2 - \frac{a_3\gamma^2}{a_5\alpha^2}(2\omega_2(\sigma+\rho) + 2a_5\alpha^2 + 3a_6\gamma^2) + 2\sigma + 2a_3\gamma^2\right]^2 + \left[a_1 + \frac{a_3\gamma^2}{a_5\alpha^2}\omega_2 a_4\right]^2 \tag{28}$$

where

$$\gamma^2 = \frac{1}{3a_6}[a_5^2\alpha^4 - \omega_2^2 a_4^2 - 2\omega_2(\sigma+\rho) - 2a_5\alpha^2] \tag{29}$$

Equation (27d) shows that in steady state the amplitude of the bending mode, α, must be such that:

$$\alpha^2 \geqslant \frac{\omega_2 a_4}{a_5} \tag{30}$$

If $\gamma = 0$, equation (28) reduces to equation (26).

Equation (29) shows that a steady state is only possible in the torsional mode, i.e. $\gamma^2 \geqslant 0$, for certain values of a_4, a_5 and $(\sigma + \rho)$.

Equations (27) were solved numerically to obtain the values of α, β, γ, and δ as functions of time. These values were then substituted into equations (19), (13), (14), (5), and (6) to yield the time histories of u and θ. Some sample time histories are shown in *Figures 9* and *10* and correspond to $\sigma = \rho = -10$. The u-response in this case decreases more rapidly than in the primary resonance case (*cf. Figure 7*) due to the energy transfer through the nonlinearities and the coupling with the θ-mode. However, the u-mode eventually grows and reaches a steady state amplitude and frequency. The steady state is exactly the same as that of the primary resonance as was proven analytically.

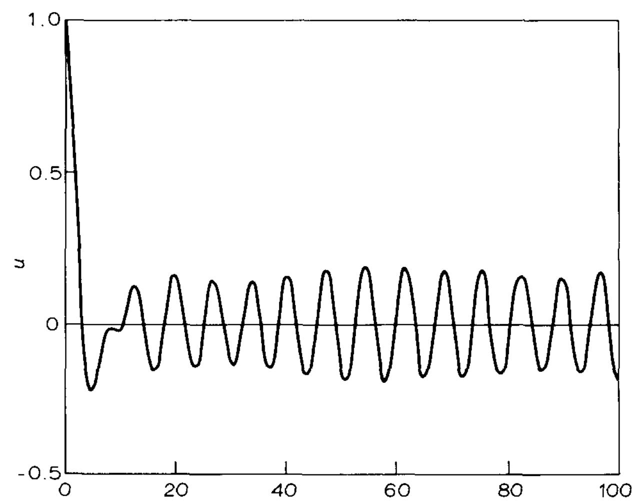

Figure 9 Bending or lateral response for non-zero ground excitation when bending frequency is approximately equal to both excitation frequency and torsion frequency

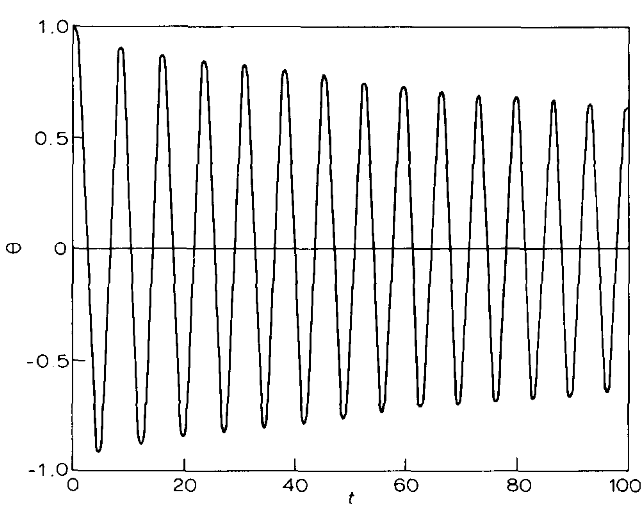

Figure 10 Torsional response for non-zero ground excitation when bending frequency is approximately equal to both excitation frequency and torsion frequency

Due to the energy transfer between the u- and o-modes the u-mode exhibits a modulation initially as shown *Figure 9*. The θ-mode is damped more slowly than in the primary resonance case, but eventually dies out as shown in *Figure 10*.

Conclusions

A multiple-scale asymptotic technique has been employed to study the torsional oscillations of a single-storey structure with cubic softening stiffness. The technique has been used to show that a single-storey structure can reach a limit cycle, i.e. a steady state solution, when the excitation frequency approaches the natural frequency of the bending mode. This case was labelled primary resonance and is deeply affected by the cubic nonlinearities of the problem. The steady state amplitude, when plotted *versus* the excitation frequency, has three possible values for a certain range of excitation frequencies. The lower and upper branches of the steady state response are stable, while the middle branch is unstable.

When the natural frequency of the bending model is near both the excitation frequency and the natural frequency of the torsional mode, the single-storey structure exhibits a limit cycle behaviour, of constant amplitude and frequency, in the bending mode. This limit cycle is exactly the same as that occurring in the case of primary resonance. The free response of the single-storey structure has also been studied, both analytically and numerically. The results show that the bending mode is damped much faster than the torsional one.

The two-time perturbation technique employed in this study identifies instability regions, provides the structural response analytically, and demonstrates that torsional oscillations in single-storey symmetrical structures are caused by torsional eccentricities. This has been mathematically proved as the differential equations for the torsional amplitude γ are homogeneous functions of γ so that torsional oscillations can only occur if $\gamma \neq 0$ initially. It has also been shown that torsional oscillations are induced by the nonlinear coupling between the lateral (bending) and torsional movements caused by the nonlinear force-displacement characteristics of the resisting elements.

Actual earthquake ground motions have a broader frequency spectrum than the simple-frequency motion used in this study. The two-time perturbation technique employed here can be readily extended to multi-frequency ground excitations. However, it may not be worth pursuing such an extension as the torsional response can be viewed as the secondary response of the system. This is because for a ground motion without a rotational component, the lateral response of a symmetrical structure can be viewed as a forced vibration which depends directly on the ground motion, while the torsional response is excited by the lateral motion whenever the latter has a large amplitude or a frequency of the order of the natural frequency of torsion.

The two-time asymptotic analysis employed in this study indicates that torsional oscillations in single-storey symmetrical structures can be reduced by decreasing the structural eccentricities and designing the structure in such a manner that the natural frequency of bending does not coincide with the natural frequency of torsion. It also shows that the cubic force-displacement relationship of the structural elements plays an important role in torsional oscillations. Large lateral displacements cause large torsional motions. Thus, sufficient ductility should be built into the resisting elements to withstand the torsional oscillations of the structure whenever the natural torsional frequency is approximately equal to the natural lateral frequency. This result is valid even when the ground motion frequency does not coincide with the natural lateral frequency as indicated in the case of free response with frequency incidence.

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